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EFFECT OF SPECULARLY REFLECTING GRAY SURFACE ON THERMAL RADIATION THROUGH A TUBE AND FROM ITS HEATED WALL

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ABSTRACT

An analysis was made of specular radiation exchange within a circular tube open at both ends and in vacuum. The tube is exposed to thermal radiation from an environment at each end and can have a uniform heat flux supplied at its wall. The external surface of the tube wall is insulated, while the internal surface is a gray, specular (mirrorlike) reflector for thermal radiation. The integral equations governing the radiation exchange are solved to determine the internal surface temperature and the amount of heat transmitted through the tube from the environment at one end to the other. Specular reflections were found to reduce the maximum surface temperature of the heated wall as compared with diffuse reflections, and in some instances the maximum temperature was below the value for a black surface. The energy transmitted through the tube was larger for the specularly reflecting wall than for diffuse reflections. It was also shown that the energy transmitted for a diffusely reflecting gray wall is the same as for a black wall.

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NOMENCLATURE

A area on inside surface of tube wall

D tube diameter

E error correction for separable kernel solution

F configuration factor for direct radiation between a ring element on the tube wall and a circular area at end of tube

G exchange factor for specular radiation between two ring elements on inside of tube

K configuration factor for direct radiation between two ring elements on inside of tube

L length of tube

l dimensionless length, L/D

M exchange factor for specular radiation between a ring element on the tube wall and a circular area at end of tube

P see Eq. (12)

Q heat transmitted through length of tube

q heat added per unit area at tube wall

T absolute temperature

X axial length coordinate measured from left end of tube

x dimensionless coordinate, X/D

x_n defined as $x/(n+1)$

y coordinate equal to $l-x$; $y_n = y/(n+1)$

ϵ emissivity of surface

θ dimensionless temperature, $\sigma T_w^4/q$

θ_e dimensionless temperature, T_w^4/T_l^4

$\bar{\theta}$ solution for θ using separable kernel (see Eq. (18))

Ξ location of ring element radiating to element at X

ξ dimensionless variable, Ξ/D

σ Stefan-Boltzmann constant

Φ see Eq. (12)

ψ integral defined in Eq. (21)

Subscripts

l environment at left end of tube

r environment at right end of tube

w tube wall

INTRODUCTION

When the exchange of thermal radiation between surfaces is being determined, it is often assumed that the surfaces are diffuse reflectors. However, as discussed in [1], many materials reflect at least partially in a specular manner. This is especially true for surfaces with a mirrorlike finish where the ~~radiation exchange~~ reflections will be almost entirely specular. The purpose of this paper is to investigate the limiting case of ~~purely specular~~ radiation exchange in an enclosure with purely specular reflections and compare it with ~~diffuse~~ the behavior for completely diffuse reflections. ~~The con-~~ In both cases it is a reasonable assumption to have the emission from the surface diffuse. The configuration which was studied is a heated cylindrical tube open at both ends. This is of interest because there is a possibility that, for a specified wall heat flux, specularly reflecting walls may diminish the wall temperature in some regions of the tube. Also with a curved polished tube it may be possible to channel heat from a source to another location not in direct view of the source. In this case, if the emissivity of the surface is low, the energy radiated into one end of the tube will be reflected down the length of the enclosure and will emerge from the other end with only slightly diminished intensity. The transmittance through a straight tube will be one of the results of the analysis.

There are a few references which are pertinent to the present study. For diffusely reflecting gray walls, temperature distributions were found for short uniformly heated tubes in [2], and the results are extended here to larger length-diameter ratios. The diffuse solutions are needed for comparison with the specular results. For a specular reflecting gray surface this configuration has been treated in [3] by an approximate method for very long tubes, but a complete solution was not obtained. For a cylindrical hole closed at one end and at constant wall temperature, the heat flux issuing from the open end for specular reflecting gray surfaces was considered in [4]. There is an error in Eq. (7) of this reference which is corrected in [5].

The cylindrical tube considered here is open to a different environment temperature at each end. The tube is in a vacuum so that heat exchanges occur only by thermal radiation. Axial heat conduction within the tube wall is neglected. A specified uniform heat input is supplied at the tube wall, and the outer surface of the tube is assumed perfectly insulated so that all of the energy received by the wall must be transferred out through the ends of the enclosure. The inside surface of the wall is assumed to be gray and to reflect specularly.

The energy equation governing the radiation exchange is a linear Fredholm integral equation. Because of the linearity, it is convenient to break the general problem into two parts, (1) uniform heating applied at the tube wall with the external environment at both ends of the tube at zero temperature, and (2) the wall unheated with the environment temperature equal to zero at one end and having a specified value at the other end. These can be combined to give the general case as shown in [2]. The integral equation is solved by two separate methods. In one the integrals are approximated in finite difference form by using Simpson's rule. This yields a set of linear algebraic equations which are solved simultaneously for the wall temperature at incremental lengths along the wall. In the second method the wall temperature inside the integral sign is expanded in a Taylor series as suggested in [3]. This transforms the integral equation into an ordinary differential equation which is solved numerically on a digital computer. These methods were applied to both the specular and diffuse cases for a range of L/D and ϵ.

ENERGY BALANCE

The energy equation for the surface temperature of the tube is found by forming a heat balance on a cylindrical element of differential area dA_X located at X on the inside surface of the tube (Fig. 1). The energy leaving the element by radiation is

$$\epsilon\ \sigma T_w^4(X)\,dA_X \tag{1}$$

The energy supplied to the element is composed of three terms. The first is the specified uniform heat flux supplied to the tube

$$q\,dA_X \tag{2}$$

The second includes the contributions from the environments at the left and right ends of the tube. It is assumed that the environments can be represented by black planes at the ends of the tube. Thus the radiation entering through the ends is both diffusely and uniformly distributed over the end openings. The exchange factor for specular radiation exchange from an element at X to the circular opening at the left end is called M(X), and it is derived later. The factor M(X) has been based on an element of tube wall area. The absorptivity of the surface is assumed constant and equal to its emissivity using the usual gray wall assumption. The radiation entering the left end is σT_l^4 per unit area. The portion of this that arrives at X and is absorbed is

$$\sigma T_l^4 \, \epsilon M(X) dA_X \qquad (3a)$$

In a similar fashion the radiation supplied by the environment at the right end is

$$\sigma T_r^4 \, \epsilon M(L-X) dA_X \qquad (3b)$$

The third energy input to the element is that supplied from the other elements of the tube wall. The exchange factor for specular radiation between two ring elements a distance Z apart is defined as G(Z) and is derived later. Then the energy radiated from one element at Ξ ~~which~~ that reaches and is absorbed by another at X is

$$\epsilon^2 \sigma T_w^4\left(\frac{\Xi}{D}\right) G\left|\frac{\Xi - X}{D}\right| dA_X d\frac{\Xi}{D} \qquad (4)$$

The contribution from the entire wall is found by integrating Eq. (4) over the length of the tube:

$$dA_X \int_0^{X/D} \epsilon^2 \sigma T_W^4\left(\frac{\Xi}{D}\right) G\left(\frac{X-\Xi}{D}\right) d\frac{\Xi}{D} + dA_X \int_{X/D}^{L/D} \epsilon^2 \sigma T_W^4\left(\frac{\Xi}{D}\right) G\left(\frac{\Xi-X}{D}\right) d\frac{\Xi}{D} \tag{5}$$

The heat balance can now be formed by equating Eq. (1) to the sum of (2), (3), and (5):

$$\epsilon \sigma T_W^4\left(\frac{X}{D}\right) = q + \epsilon \sigma T_l^4 \, M\left(\frac{X}{D}\right) + \epsilon \sigma T_r^4 \, M\left(\frac{L-X}{D}\right)$$

$$+ \int_0^{X/D} \epsilon^2 \sigma T_W^4\left(\frac{\Xi}{D}\right) G\left(\frac{X-\Xi}{D}\right) d\frac{\Xi}{D} + \int_{X/D}^{L/D} \epsilon^2 \sigma T_W^4\left(\frac{\Xi}{D}\right) G\left(\frac{\Xi-X}{D}\right) d\frac{\Xi}{D} \tag{6}$$

The expressions for M and G will now be derived.

Exchange factor between ring element and end of tube. - For specular radiation the exchange factor M is composed of the sum of exchanges due to direct radiation, one reflection, two reflections, and so forth. The factor for direct exchange between a ring element at X and a circular opening at the left end of the tube has been given in [6] or [2] by

$$F(X) = \frac{\left(\frac{X}{D}\right)^2 + \frac{1}{2}}{\left[\left(\frac{X}{D}\right)^2 + 1\right]^{1/2}} - \frac{X}{D}$$

The energy exchanged by one reflection can be computed as shown in Fig. 2. The radiant energy from the element at X that passes directly through the plane at X/2 will either leave the tube opening directly, or the part that is reflected will leave the opening after one reflection. The radiant energy from the element at X that strikes the wall before passing through the plane at X/2 will require two or more reflections before leaving the tube.

As a result, the exchange between the ring element and the end of the tube for one reflection equals the reflectivity of the wall multiplied by the difference between the energy radiated through the plane at $X/2$ and the energy radiated directly out of the tube. The configuration factor for the energy that is radiated through $X/2$ is given by $F(X/2)$, and the factor for the energy leaving directly is given by $F(X)$. Consequently the exchange factor for one reflection is equal to $(1-\epsilon)\,[F(X/2) - F(X)]$.

In a similar fashion if there are two reflections the factor is found as follows. The radiant energy from the element at X that strikes the wall before passing through a plane located a distance $X/3$ from X will require three or more reflections for any part of it to leave the tube, while the energy that passes directly through a plane at $X/2$ will require one or no reflections before any part of it leaves. For two reflections the exchange factor is then the reflectivity squared times the difference between the configuration factor for the energy passing through a cross section at a distance $X/3$ from X and the configuration factor for the energy passing through the cross section at $X/2$:

$$(1-\epsilon)^2 \left[F\left(\frac{X}{3}\right) - F\left(\frac{X}{2}\right)\right]$$

In a similar fashion for n reflections

$$(1-\epsilon)^n \left[F\left(\frac{X}{n+1}\right) - F\left(\frac{X}{n}\right)\right]$$

The exchange factor between a ring and the left end of the tube is found by summing the contributions from all the reflections:

$$M(X) = F\ (X) + \sum_{n=1}^{\infty} (1-\epsilon)^n \left[F\left(\frac{X}{n+1}\right) - F\left(\frac{X}{n}\right) \right] \qquad (7a)$$

The factor from a ring to the right end of the tube follows in an identical fashion:

$$M(L-X) = F(L-X) + \sum_{n=1}^{\infty} (1-\epsilon)^n \left[F\left(\frac{L-X}{n+1}\right) - F\left(\frac{L-X}{n}\right) \right] \qquad (7b)$$

Exchange factor between two ring elements. - Here again the total exchange is computed by summing the direct exchange, exchange by one reflection, exchange by two reflections, and so forth. The direct exchange is obtained from the configuration factor between two ring elements separated by a distance Z (see [6] or [2])

$$K(Z) = 1 - \frac{\left(\frac{Z}{D}\right)^3 + \frac{3Z}{2D}}{\left[\left(\frac{Z}{D}\right)^2 + 1\right]^{3/2}} \qquad \frac{Z}{D} \geq 0$$

The energy arriving after one reflection at an element located Z distance away from the emitting element is obtained as follows (see Fig. 3). The only energy from dA_Ξ that can reach dA_X after one reflection must be reflected from an element $dA_{Z/2}$ halfway between dA_Ξ and dA_X. The exchange factor for energy leaving dA_Ξ which arrives directly at the element $dA_{Z/2}$ is $K\left(\frac{Z}{2}\right)$. An amount $(1-\epsilon)$ of this is specularly reflected to dA_X. However, as shown in Fig. 3, only half of the reflected radiation is intercepted by the element dA_X, since the radiation from any point on dA_{Ξ} that is reflected at $Z/2$ will irradiate a region twice as large as dA_X when it reaches the X location. To account for this, a factor of 1/2 has to be introduced, and the exchange factor for one reflection becomes

$$\frac{(1-\epsilon)}{2} K\left(\frac{Z}{2}\right)$$

In a similar fashion for n reflections the result is

$$\frac{(1-\epsilon)^n}{n+1} K\left(\frac{Z}{n+1}\right)$$

The total exchange is found by summing the results for all reflections to give

$$G(Z) = K(Z) + \sum_{n=1}^{\infty} \frac{(1-\epsilon)^n}{n+1} K\left(\frac{Z}{n+1}\right) \tag{8}$$

The exchange factors given by Eqs. (7) and (8) can now be substituted into the energy equation (6), which is to be solved for the unknown wall temperature.

SUPERPOSITION OF SOLUTIONS

The energy Eq. (6) is linear in the variable T_w^4 and as a consequence the general solution can be simplified by considering two more elementary solutions which can be combined to yield results for any combination of imposed wall heat flux q, left environment temperature T_l, and right environment temperature T_r. One of the basic solutions is where the wall is heated but the environments are maintained at zero temperature. In this case Eq. (6) reduces to

$$\epsilon\theta(x) = 1 + \epsilon^2 \int_0^x \theta(\xi)G(x-\xi)d\xi + \epsilon^2 \int_x^l \theta(\xi)G(\xi-x)d\xi \qquad (9)$$

where $\theta = \frac{\sigma T_W^4}{q}$, $x = \frac{X}{D}$, and $\xi = \frac{\Xi}{D}$.

The second basic solution is where no heat is supplied at the channel walls and the right environment T_r is maintained at zero temperature while the left environment T_l has a specified value. Equation (6) then reduces to

$$\theta_e(x) = M(x) + \epsilon \int_0^x \theta_e(\xi)G(x-\xi)d\xi + \epsilon \int_x^l \theta_e(\xi)G(\xi-x)d\xi \qquad (10)$$

where

$$\theta_e = \frac{T_W^4}{T_l^4}$$

When θ and θ_e are known, the general solution for any q, T_l and T_r is found from

$$\sigma T_W^4(x) = q\theta(x) + \sigma T_l^4\theta_e(x) + \sigma T_r^4\theta_e(l-x) \qquad (11)$$

The quantities $\theta_e(x)$ and $\theta_e(l-x)$ can be related by noting that, for $q=0$ and $T_l^4 = T_r^4$, T_W^4 must equal T_l^4, and Eq. (11) gives $\theta_e(l-x) = 1-\theta_e(x)$. As a result Eq. (11) can be written in the alternate form

$$\sigma T_W^4(x) = q\theta(x) + \sigma(T_l^4-T_r^4)\theta_e + \sigma T_r^4 \qquad (11a)$$

We now proceed to obtain solutions for θ and θ_e from Eqs. (9) and (10).

NUMERICAL SOLUTION OF INTEGRAL EQUATION

It is convenient to write Eqs. (9) and (10) in the same form. This is given as

$$\Phi(x) = P(x) + \epsilon \int_0^x \Phi(\xi)G(x-\xi)d\xi + \epsilon \int_x^l \Phi(\xi)G(\xi-x)d\xi \qquad (12)$$

where when

$$\begin{cases} P(x) = 1 \\ \Phi(x) = \epsilon\theta(x) \end{cases} \qquad \begin{cases} P(x) = M(x) \\ \Phi(x) = \theta_e(x) \end{cases}$$

The numerical solution of Eq. (12) was found by dividing the tube into increments by selecting $N + 1$ points along its length, where N is an even number. The incremental length between points is then $\Delta = l/N$. Equation (12) is applied at each point, and the integrals are approximated by using Simpson's rule. This gives a set of $N + 1$ linear algebraic equations which can be solved simultaneously for the unknown temperatures along the tube. There is a special approximation that must be made because the integrand $\Phi(\xi)G|x-\xi|$ has a discontinuity in derivative at $\xi = x$. Simpson's rule is based on fitting a parabola between three points, and if the discontinuity is anywhere between the end points, the approximation to the curve may be poor. In these cases, since the function Φ does not have a discontinuity, an intermediate value of Φ is interpolated at $\xi = x-\frac{\Delta}{2}$ halfway between the discontinuity $\xi = x$ and the previous point $\xi = x-\Delta$. Then Simpson's rule is applied for the three points $x-\Delta$, $x - \frac{\Delta}{2}$, and x. The value of Φ at $x - \frac{\Delta}{2}$ must be found in terms of Φ at the surrounding points so as not to introduce any new unknowns. This was done by using the interpolation formula

$$\Phi\left(x - \frac{\Delta}{2}\right) = \frac{3}{8}\Phi\ (x-\Delta) + \frac{3}{4}\Phi\ (x) - \frac{1}{8}\Phi\ (x+\Delta)$$

The same procedure was used to approximate the integral in the interval between $\xi = X$ and $X + \Delta$. Further details on this procedure are given in [7]. When solving the set of simultaneous equations the computations are considerably shortened by noting as discussed in [2] for the diffuse case that θ is symmetric and θ_e is skew symmetric about $l/2$.

TAYLOR SERIES SOLUTION

A shortcoming of the numerical method described in the preceding section is that, when the tube is long, large numbers of increments are required, and it becomes difficult to obtain an accurate solution to a large set of simultaneous equations. Hence an alternate procedure was investigated which has been suggested in [3]. In this method a Taylor series approximation was made for the temperature function $\Phi(\xi)$ in Eq. (12):

$$\Phi(\xi) = \Phi(x) + (\xi - x)\left[\frac{d\Phi}{dx}\right]_x + \frac{(\xi - x)^2}{2}\left[\frac{d^2\Phi}{dx^2}\right]_x + \ldots \tag{13}$$

In the present study the series expansion was not carried beyond the second derivative term.

Substituting Eq. (13) for $\Phi(\xi)$ in the integrals in Eq. (12) gives

$$\Phi(x) = P(x) + \epsilon \int_0^l \left\{\Phi(x) + (\xi - x)\left[\frac{d\Phi}{dx}\right]_x + \frac{(\xi - x)^2}{2}\left[\frac{d^2\Phi}{dx^2}\right]_x\right\} G\ |\xi - x|\ d\xi \tag{14}$$

The integration can now be carried out because it is with respect to ξ, while Φ and its derivatives are functions of x and hence can be taken out of the integral. This gives the following second-order differential equation:

$$\frac{\epsilon}{2}\frac{d^2\Phi}{dx^2}\left[\int_0^{l}(\xi-x)^2\,G|\xi-x|\,d\xi\right]+\epsilon\frac{d\Phi}{dx}\left[\int_0^{l}(\xi-x)\,G|\xi-x|\,d\xi\right]$$

$$+\,\Phi\left[\epsilon\int_0^{l}G|\xi-x|\,d\xi-1\right]=-P(x) \qquad (15)$$

where the terms in the brackets are given by

$$\int_0^{l}(\xi-x)^2\,G|\xi-x|\,d\xi=\frac{1}{3}\sum_{n=0}^{\infty}(1-\epsilon)^n\,(n+1)^2\left[x_n^3+y_n^3+2-\frac{x_n^4+\frac{x_n^2}{2}+1}{(x_n^2+1)^{1/2}}-\frac{y_n^4+\frac{y_n^2}{2}+1}{(y_n^2+1)^{1/2}}\right]$$

$$\int_0^{l}(\xi-x)\,G|\xi-x|\,d\xi=\frac{1}{2}\sum_{n=0}^{\infty}(1-\epsilon)^n\,(n+1)\left[y_n^2-x_n^2+\frac{x_n^3}{(x_n^2+1)^{1/2}}-\frac{y_n^3}{(y_n^2+1)^{1/2}}\right]$$

$$\int_0^{l}G|\xi-x|\,d\xi=\sum_{n=0}^{\infty}(1-\epsilon)^n\left[1+\frac{l}{n+1}-\frac{x_n^2+\frac{1}{2}}{(x_n^2+1)^{1/2}}-\frac{y_n^2+\frac{1}{2}}{(y_n^2+1)^{1/2}}\right]$$

For the Taylor series approximation to yield accurate solutions, one of two conditions must be fulfilled: (1) The kernel $G|\xi-x|$ decreases sufficiently fast as $|\xi - x|$ increases from zero that over the interval where the Taylor series approximation for Φ is no longer accurate very little is contributed to the integrals, or (2) The solution $\Phi(x)$ is such a simple curve that it can be represented with good accuracy over its entire length by a Taylor series of three terms expanded about any point. The differential equation (15) was solved by forward integration using the Runge-Kutta method on a digital computer. The boundary conditions are different for the θ and θ_e functions, and will be discussed one at a time.

The θ function is symmetric about the center of the tube $(x = l/2)$ and hence the integration can be started there with the boundary condition

$$\frac{d\theta}{dx} = 0 \text{ at } x = l/2$$

The integration is carried forward to $x = l$. To start the integration a value of $\theta(l/2)$ has to be guessed, and for each value that is chosen a different temperature distribution will be obtained. To determine which solution is correct the distributions are each tested in an overall heat balance which will be satisfied when the correct $\theta(l/2)$ is used. The heat balance is derived as follows. The heat added at the tube wall is $q\pi DL$. The heat radiated out of the left end of the tube is

$$\pi D \int_0^L \epsilon\sigma T_W^4(X) M(X)\, dX$$

Since by symmetry the heat leaving the right side of the tube is equal to that leaving the left side, the heat balance can be written as

$$\frac{q\pi DL}{2} = \pi D \int_0^L \epsilon\sigma T_W^4(X) M(X)\, dX \qquad (16)$$

This can be placed in the dimensionless form

$$\frac{l}{2} = \int_0^l \epsilon\,\theta(x) M(x)\, dx \tag{16a}$$

The correct Taylor series solution for θ will satisfy this relation

The function θ_e is skew symmetric about $l/2$ and in this case the integration is started from the boundary condition

$$\theta_e = 1/2 \quad \text{at} \quad x = l/2$$

The first derivative $d\theta_e/dx$ at $l/2$ is guessed, and the differential equation is integrated to $x = l$. The solution is then tested in the boundary condition obtained by evaluating the original integral equation (10) at $x = l$

$$\theta_e(l) = M(l) + \epsilon \int_0^l \theta_e(\xi)\; G(l-\xi)\; d\xi$$

If this is not satisfied a new derivative at $l/2$ is tried and the solution repeated. Since the differential equations for θ and θ_e are linear, only two trial solutions are needed to interpolate the correct solution.

ERROR CORRECTION SOLUTION

For comparison with the results for specular reflections it is necessary to have solutions for the case where the surface is diffuse. As shown in [2] these can be found when the result for a black tube is known. For a black wall, solutions can be found numerically or with the Taylor series method by letting $\epsilon = 1$ in the previous formulations. Results for short tubes are also given in [2] where a variational method and a separable kernel method were employed. The seperable kernel method begins to be in error for tubes with $L/D > 5$, and hence the analysis in [2] needs to be extended to apply for longer tubes before it can be compared with the present results which extend to $L/D = 20$. This can be done by finding an error correction for the separable kernel solution by using a method outlined in [8]. This discussion is limited to determining the solution for θ with a black wall. For this case the integral equation (9) reduces to

$$\theta(x) = 1 + \int_0^x \theta(\xi)K(x-\xi)d\xi + \int_x^l \theta(\xi)K(\xi-x)d\xi \qquad (17)$$

An approximate solution $\bar{\theta}$ found in [2] by using the approximate separable kernel is

$$\bar{\theta} = l + 1 + 2\ (xl - x^2) \qquad (18)$$

A corrected solution θ is found by determining the error $E(x)$ introduced by using the approximate kernel so that

$$\theta(x) = \bar{\theta}(x) + E(x) \qquad (19)$$

The error is found from the integral equation

$$E(x) = \psi(x) + \int_0^l K|x-\xi|\ E(\xi)d\xi \qquad (20)$$

where

$$\psi(x) = \int_0^l \left(K|x-\xi| - e^{-2|x-\xi|}\right)\bar{\theta}(\xi)d\xi \qquad (21)$$

The quantity in parenthesis is the difference between the exact and approximate kernels.

Equation (20) is of the same type as the original Eq. (17) and can be solved by using an approximate kernel to yield an approximate $E(x)$.

The error in $E(x)$ will be a small second-order difference in the final solution. Introducing the separable kernel into the integral involving $E(\xi)$ in Eq. (20) gives

$$E(x) = \psi(x) + \frac{1}{e^{2x}} \int_0^x E(\xi)e^{2\xi}d\xi + e^{2x} \int_x^{l} E(\xi)e^{-2\xi}d\xi \qquad (22)$$

By differentiating twice and subtracting the original equation multiplied by four, the integrals are eliminated to give

$$\frac{d^2E}{dx^2} = \frac{d^2\psi}{dx^2} - 4\psi$$

After integrating twice, the solution is

$$E(x) = \psi(x) - 4 \int^{\cancel{x}}\int^{\cancel{x}} \psi(x)dxdx + C_1x + C_2 \qquad (23)$$

where C_1 and C_2 are arbitrary constants which remain to be determined. The function $\psi(x)$ in Eq. (23) is found by carrying out the integration in Eq. ($\cancel{22}$ 21). It is substituted into Eq. (23) and integrated twice. The constant C_1 is evaluated by using the fact that $E(x)$ is symmetric about $x = l/2$ so that dE/dx at $l/2$ equals zero. The constant C_2 is found by applying the boundary condition obtained by evaluating Eq. (22) at $x = 0$. These steps require considerable algebraic manipulation which it was felt not worthwhile to include here. Hence we go directly to the final answer

$$E(x) = [1 + 2x(l-x)]\left(-\frac{1}{3} + l + l^2 + l^3/3\right) + S(x) + S(l-x) + C_2 \qquad (24)$$

where

$$S(x) = \left(\frac{l}{2} - x\right)\left[\log_e\left(x + \sqrt{x^2+1}\right)\right]$$

$$+ (x^2+1)^{-1/2}\left(\frac{41}{30} + \frac{l}{6} - \frac{lx}{6} + \frac{8}{5}x^2 + \frac{lx^2}{3} + \frac{lx^3}{6} + \frac{3x^4}{5} + \frac{2lx^4}{3} + \frac{lx^5}{3} - \frac{2x^6}{15}\right)$$

and

$$C_2 = \frac{2}{(1+e^{-2l})}\left[\int_0^l \left\{S(x)+S(l-x) + \left[1+2x(l-x)\right]\right.\right.$$

$$\left.\left.\left[-\frac{1}{3} + l + l^2 + l^3/3\right]\right\} e^{-2x}\,dx - \frac{6}{5} - \frac{2}{3}l\right.$$

$$+ \frac{l}{2}\log_e\left(l + \sqrt{l^2+1}\right) - \left(1+l^2\right)^{-1/2}$$

$$\left.\left(\frac{6}{5} + \frac{2l}{3} + \frac{21l^2}{10} + \frac{4l^3}{3} + \frac{11l^4}{10} + \frac{2l^5}{3} + \frac{l^6}{5}\right)\right]$$

The final corrected solution is obtained by adding $E(x)$ to the approximate solution given in Eq. (18). Numerical results are given in the next section.

RESULTS FOR UNIFORMLY HEATED WALLS WITH ENVIRONMENT AT ZERO TEMPERATURE

In this section results are given for the case when the tube wall is uniformly heated and the environment at each end of the tube is at zero temperature.

Diffuse surface. - For comparison with the results for a specular surface the solutions for the diffuse case are needed. In [2] it was shown that the diffuse gray solution can be obtained from the results for a black wall by using the relation

$$\sigma T_w^4 = q\left(\frac{1}{\epsilon} - 1\right) + \left(\sigma T_w^4\right)_{\epsilon=1} \tag{25}$$

Since the black wall solution does not depend on whether the surface is specular or diffuse, it is obtained from the present specular analysis by letting ϵ go to unity. For black tubes with lengths of $l = 5$, 10, and 20, values of $T_w\left(\frac{\sigma}{q}\right)^{1/4}$ are tabulated in Table 1 for both the direct numerical computation and the Taylor series method. Included for comparison are the solutions from [2] obtained by using a variational procedure and an approximate separable kernel. The latter method gives less accuracy as the length of the tube increases, but after adding the error correction from the previous section the results are very good. The present solutions all agree within a fraction of 1 percent and were used along with Eq. (25) to determine the diffuse curves in Fig. 4.

Specular surface. - For specular walls results were found for a few different emissivities by both the numerical and Taylor series methods. A comparison of the results is given in Table 2 for a few different lengths, and the agreement is within 3 percent. Figure 4 compares the specular and diffuse wall temperature distributions for $l = 5$, 10, and 20. In contrast with the diffuse solutions the specular temperatures do not increase monotonically as the emissivity is reduced but go through a minimum value when the emissivity is in the range of 0.1 to 0.5. The presence of a minimum can be interpreted physically as follows. Since the outside of the heated tube is assumed perfectly insulated all the heat supplied must be dissipated through the open ends of the tube. The temperatures in the central portion of the tube will be minimized when the optimum communication is achieved between this part of the tube and the end openings. The mirrorlike reflections of a specular surface are quite effective in transmitting energy through the tube and can even be more effective than a black surface. Figure 5 shows the wall temperature at the midpoint of the tube ($X = L/2$) as a function of emissivity. As the tube length is increased, a smaller emissivity is required to obtain the maximum reduction of midpoint temperature, and the temperatures are substantially below the values for a black surface.

RESULTS FOR UNHEATED WALL AND SPECIFIED ENVIRONMENT TEMPERATURES

In this section results are given for the case where no heat flux is imposed at the tube wall and the right end environment is at zero temperature while the left end environment is at a specified temperature.

Diffuse solution. - When there is no heat flux imposed at the tube wall ($q = 0$) the diffuse solution is found from Eq. (25) as

$$T_w^4 = \left(T_w^4\right)_{\epsilon=1} \tag{26}$$

This shows that, for the unheated wall, ($q = 0$), the emissivity of the wall has no influence on the wall temperature distribution. This arises from the fact that, for an externally insulated wall with zero heat conduction, at steady state all absorbed energy must be reemitted from the same location at which it was absorbed. Both the reemitted and reflected energy are diffuse, and as a result all of the heat incident upon an elemental area will leave diffusely. Hence it will not make any difference to what extent the energy leaving the surface is composed of energy that has been directly reflected or absorbed and reradiated and is thus independent of the emissivity.

Specular surface. - The results for the specular surface were carried out using both the numerical procedure and the Taylor series method outlined previously, and the two methods were in agreement within a few percent. In Fig. 6 are shown the dimensionless wall temperature distributions for different tube lengths. The curves are given for only the first half of the tube, since the results for the remaining half can readily be obtained from the skew symmetry of the solution

$$\theta_e(x) = 1 - \theta_e(l-x)$$

The curves show there is a decrease in the maximum wall temperature for specular surfaces as ϵ becomes smaller. The specular distributions are more uniform than the diffuse curves which remain the same for all ϵ. In the limit as $\epsilon \to 0$ the specular temperature distribution can be obtained from Eqs (10) and (7a) as

$$\Big(\theta_e(x)\Big)_{\epsilon=0} = \Big(M(x)\Big)_{\epsilon=0} = F(x) + \sum_{n=1}^{\infty}\left[F\left(\frac{x}{n+1}\right) - F\left(\frac{x}{n}\right)\right] = F(0) = \frac{1}{2}$$

In practice as $\epsilon \to 0$, the radiation exchange would be very small; consequently the tube would be influenced more by heat transfer from the external environment, since it would not be possible to insulate the outside surface of the wall perfectly.

A quantity which can be computed from the temperature distribution is the net heat transferred axially through the tube from the hot left environment to the right environment at zero temperature. At steady state the heat leaving the right end is equal to that entering the left end, which can be obtained as follows. The heat radiated in from the left environment is $\sigma T_l^4 \frac{\pi D^2}{4}$. The heat radiated out to the left environment from the tube wall is $\epsilon\sigma\pi D \int_0^L T_w^4(X)M(X)dX$. The net heat transferred through the tube is then

$$\frac{Q_{specular}}{\sigma T_l^4 \frac{\pi D^2}{4}} = 1 - 4\epsilon \int_0^l \theta_e M(x)dx \qquad (27)$$

The result for the diffuse wall is the same as for the black wall and can be obtained from Eq. (27) when $\epsilon = 1$. This has been plotted in Fig. 7(a) as a function of tube length. The maximum transmission occurs as the length approaches zero. For an L/D of 10 the heat transferred through a tube with a black or diffuse reflecting wall has been reduced to 10 percent of the maximum, and decreases slowly as the length is further increased.

Figure 7(b) shows how the heat transmitted through the tube is increased when the wall is specular. When the emissivity of the surface is zero the heat transmitted becomes $\sigma T_l^4 \frac{\pi D^2}{4}$ and is independent of the channel length. Thus it may be possible to transfer heat over a considerable distance by using a polished tube, and the wall temperature of the channel can be kept reasonably low by using very small emissivities. This is analogous to the phenomenon in fiber optics in which light is trapped inside a glass filament and is transmitted along its length.

CONCLUDING REMARKS

The temperature distribution along the length of a heated tube has been determined when the internal heat exchange is by radiation only and the outside of the tube is insulated. The limiting case was examined where the internal tube surface is a completely specular reflector and the results were compared with those for a diffuse surface. For a given heat input to the tube, a specular surface produces a temperature distribution which is more uniform along the tube length and in some cases reduces the maximum temperature as compared with the black or diffusely reflecting case. When a heat flux is not imposed at the tube wall, but the environment at one end of the tube is hot while the other is cold, then heat is transmitted longitudinally through the tube. For a diffuse wall the transmitted heat is not a function of wall emissivity, whereas for a specular wall the transmission is substantially increased as the emissivity is reduced. In all instances more heat is transferred through the tube in the specular reflecting case than in the black or diffuse cases, and the maximum wall temperatures are lower for the specular cases.

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TABLE I. - COMPARISON OF SOLUTIONS FOR DIMENSIONLESS WALL TEMPERATURE $(\sigma/q)^{1/4}\ T_w$ IN A HEATED BLACK TUBE ($\epsilon = 1$) WITH ZERO ENVIRONMENT TEMPERATURE.

$\frac{L}{D}$	X/L	$(\sigma/q)^{1/4}\ T_w$				
		Numerical (matrix solution)	Taylor series	Variational	Separable kernel	Error correction
5	0	1.557	1.560	1.562	1.565	1.557
	.25	1.980	1.979	1.978	1.980	1.980
	.5	2.069	2.070	2.072	2.074	2.070
10	0	1.795	1.804	1.816	1.821	1.793
	.25	2.591	2.590	2.589	2.639	2.587
	.5	2.733	2.734	2.739	2.795	2.728
20	0	2.086	2.108	2.134	2.141	2.087
	.25	3.472	3.489	3.488	3.616	3.467
	.5	3.687	3.706	3.714	3.856	3.677

TABLE II. - COMPARISON OF SPECULAR SOLUTIONS FOR DIMENSIONLESS WALL TEMPERATURE $(\sigma/q)^{1/4}\ T_W$ IN A HEATED TUBE WITH ZERO ENVIRONMENT TEMPERATURE.

$\frac{L}{D}$	X/L	$(\sigma/q)^{1/4}\ T_W$			
		$\epsilon = 0.1$		$\epsilon = 0.5$	
		Numerical (matrix solution)	Taylor series	Numerical (matrix solution)	Taylor series
5	0	2.038	2.048	1.629	1.646
	.25	2.133	2.132	1.895	1.892
	.5	2.155	2.156	1.952	1.952
10	0	2.172	2.198	1.826	1.865
	.25	2.357	2.354	2.326	2.321
	.5	2.396	2.397	2.418	2.418
20	0	2.357	2.412	2.082	2.148
	.25	2.701	2.696	2.966	2.962
	.5	2.766	2.767	3.111	3.111

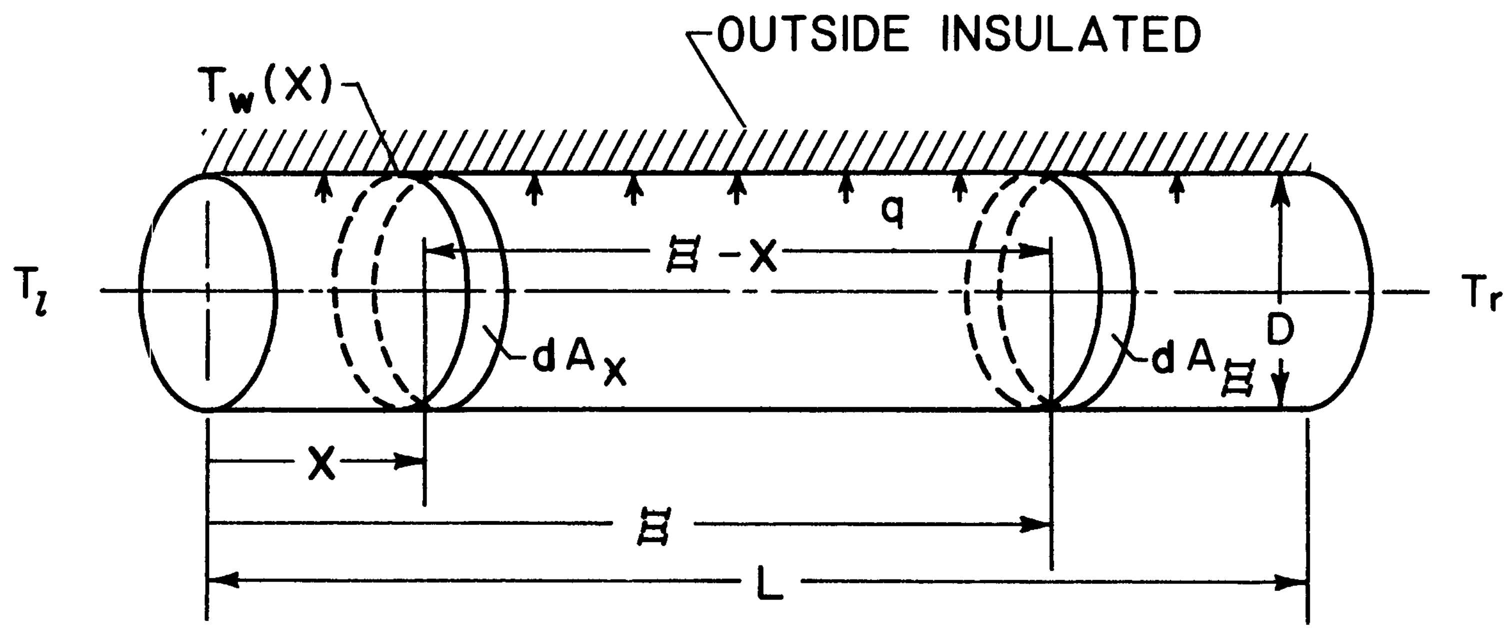

Figure 1. - Circular tube open at both ends.

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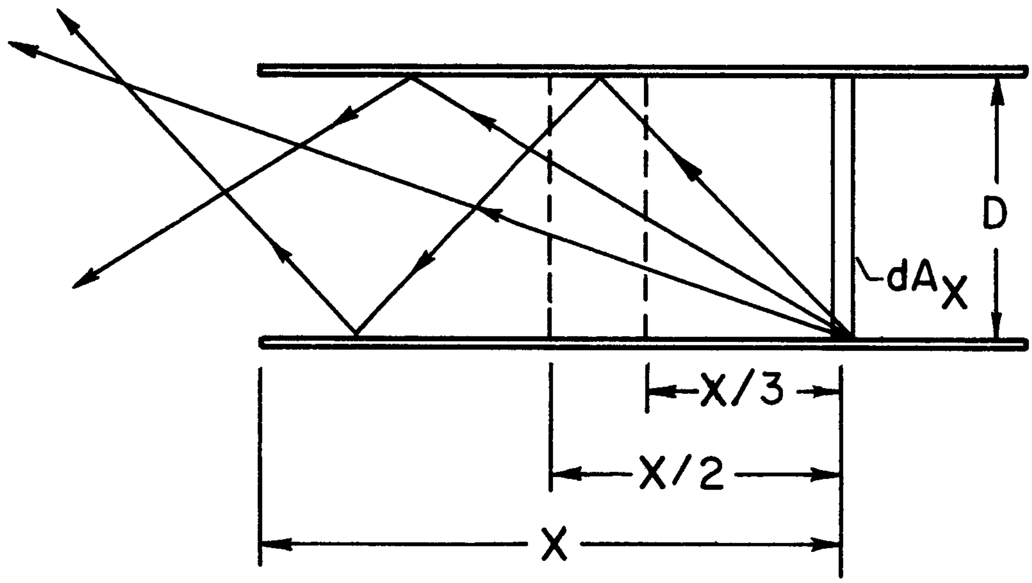

Figure 2. - Reflections of radiant beams passing from element at X to opening at left end of tube.

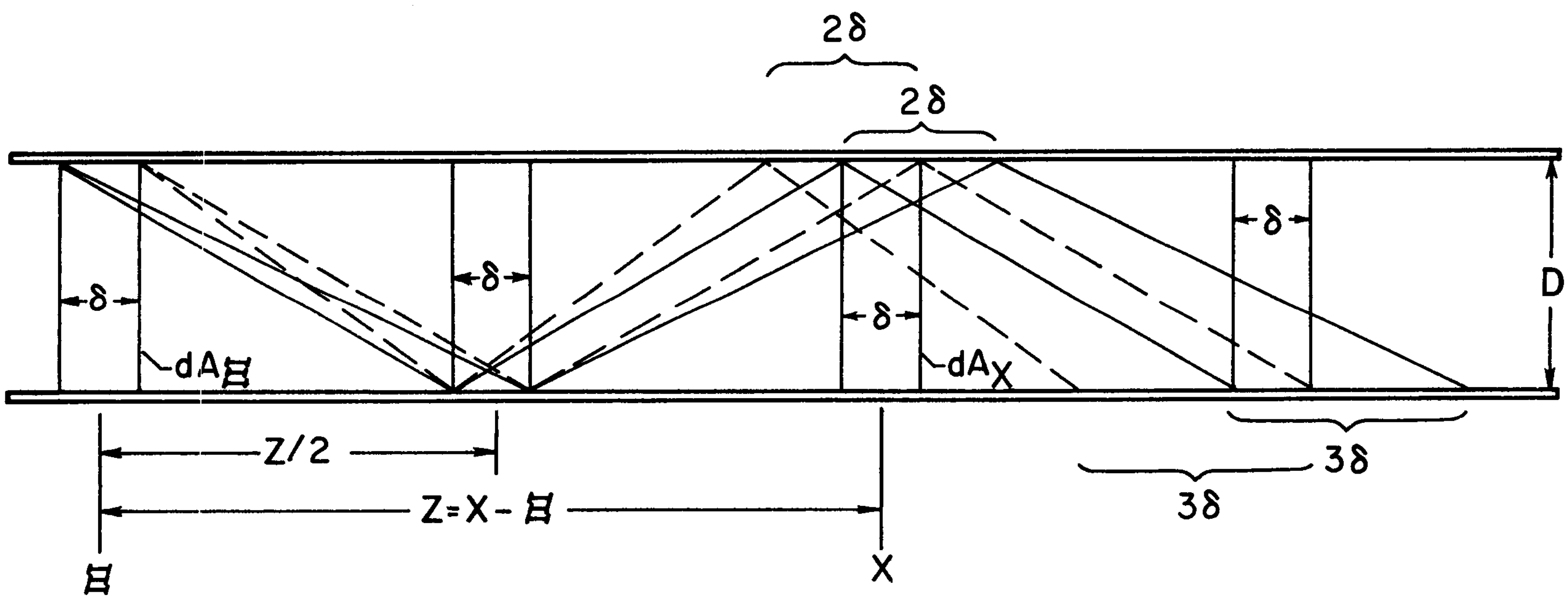

Figure 3. - Spreading of radiant beam during reflection between wall elements.

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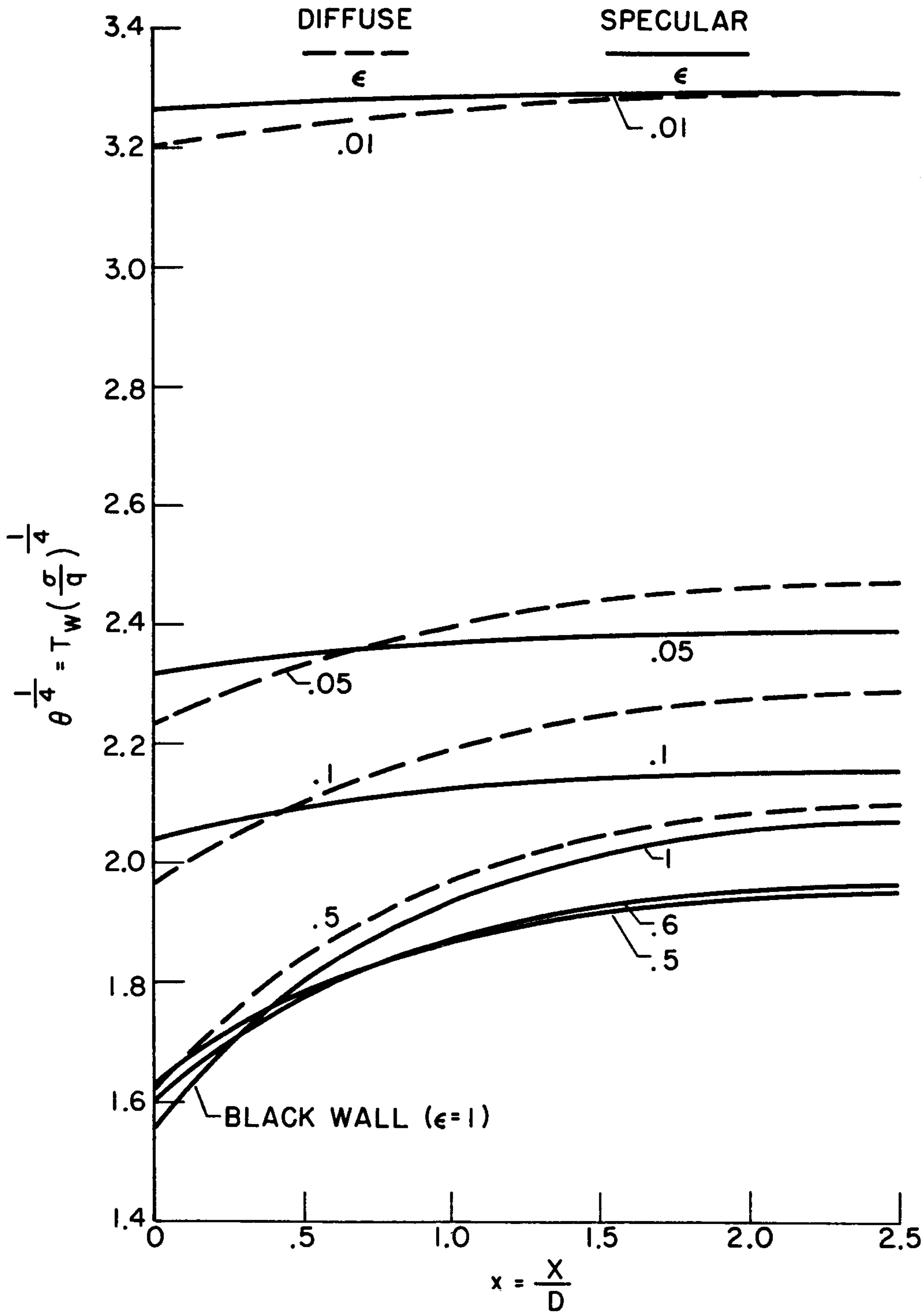

(a) L/D=5. FOR $2.5 < x \leq 5$, $\theta^{\frac{1}{4}}(x) = \theta^{\frac{1}{4}}(5-x)$.

Figure 4. - Dimensionless wall temperature distributions for specular or diffuse reflections in a heated tube. $T_l = 0$; $T_r = 0$.

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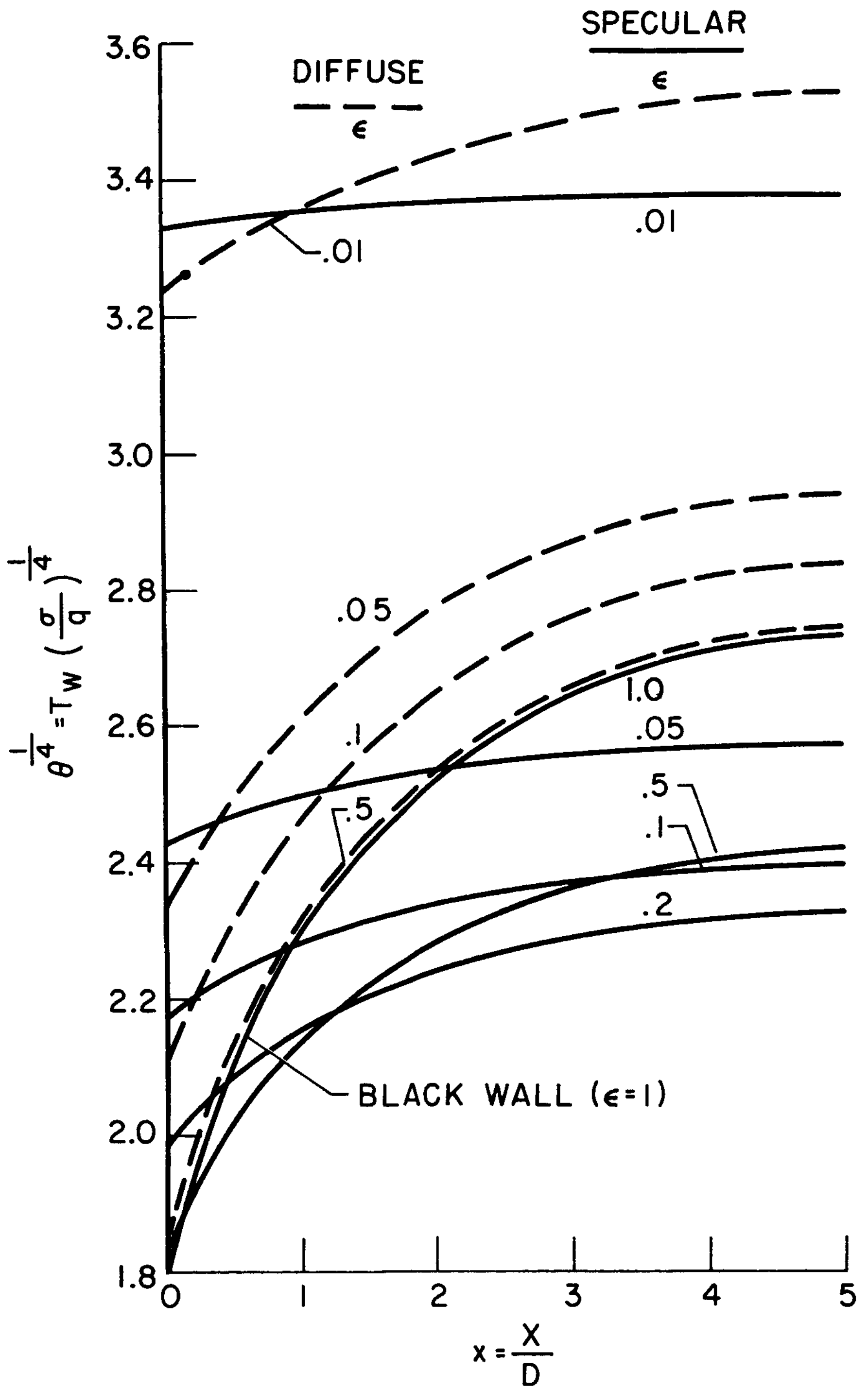
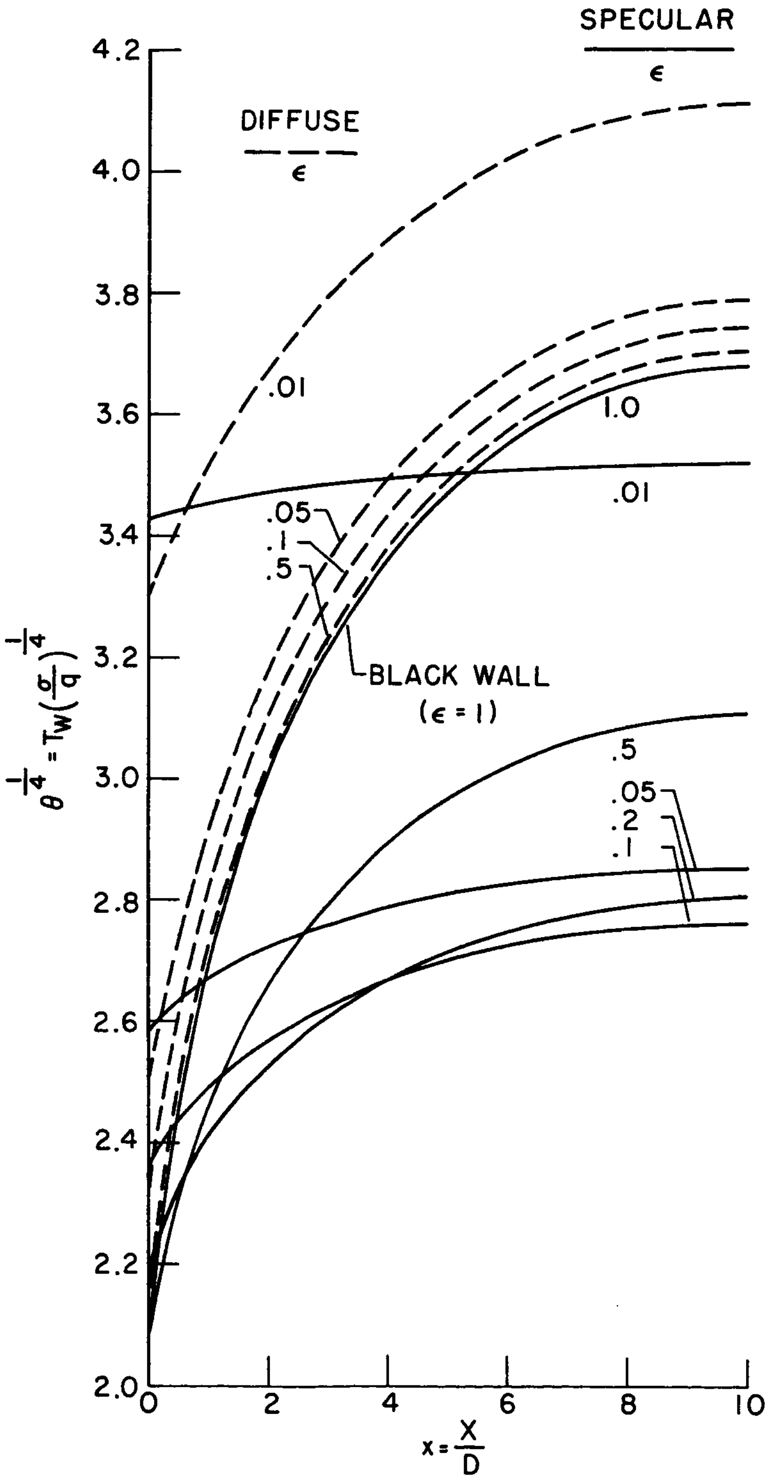


(b) L/D=10. FOR $5 < x \leq 10$, $\theta^{\frac{1}{4}}(x) = \theta^{\frac{1}{4}}(10-x)$.

Figure 4. - Continued. Dimensionless wall temperature distributions for specular or diffuse reflections in a heated tube. $T_l = 0$; $T_r = 0$.



(c) $\frac{L}{D} = 20$. FOR $10 < x \leq 20$, $\theta^{\frac{1}{4}}(x) = \theta^{\frac{1}{4}}(20-x)$.

Figure 4. - Concluded. Dimensionless wall temperature distributions for specular or diffuse reflections in a heated tube. $T_l = 0$; $T_r = 0$.

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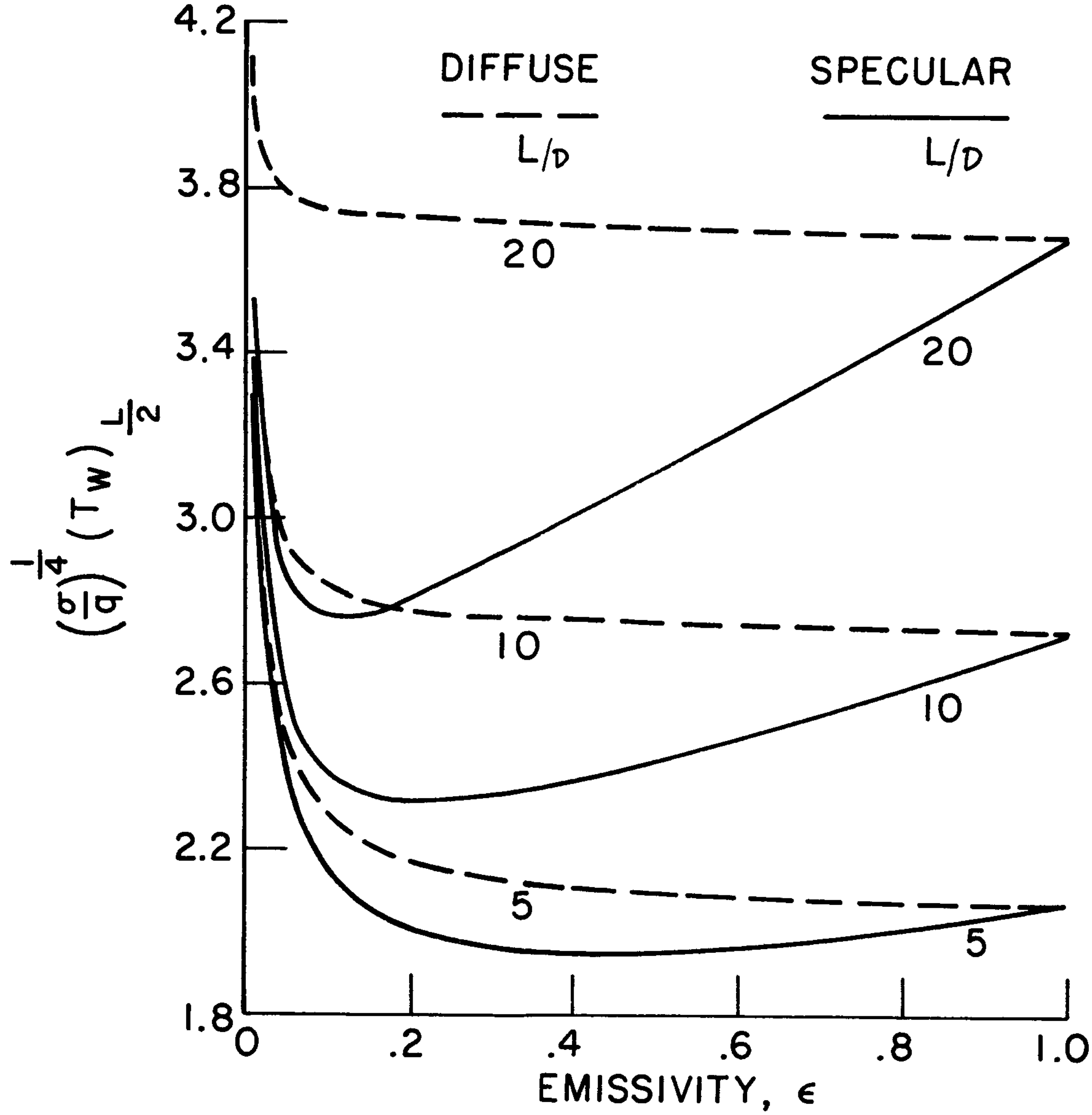

Figure 5. - Wall temperature at midpoint ($X = L/2$) of a heated tube for diffuse or specular walls. $T_l = 0$; $T_r = 0$.

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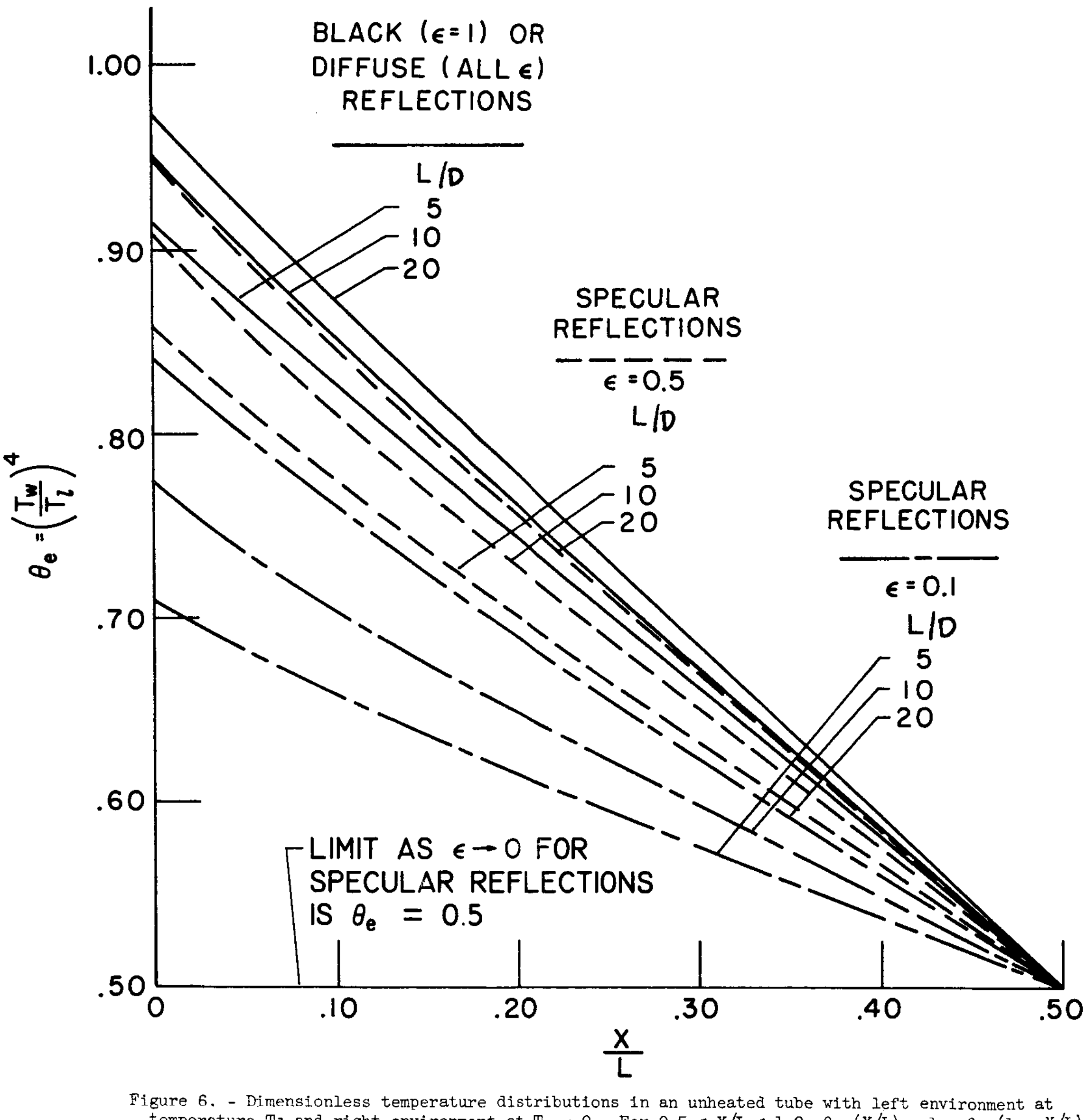

Figure 6. - Dimensionless temperature distributions in an unheated tube with left environment at temperature T_l and right environment at $T_r = 0$. For $0.5 < X/L < 1.0$, $\theta_e(X/L) = 1 - \theta_e(1 - X/L)$.

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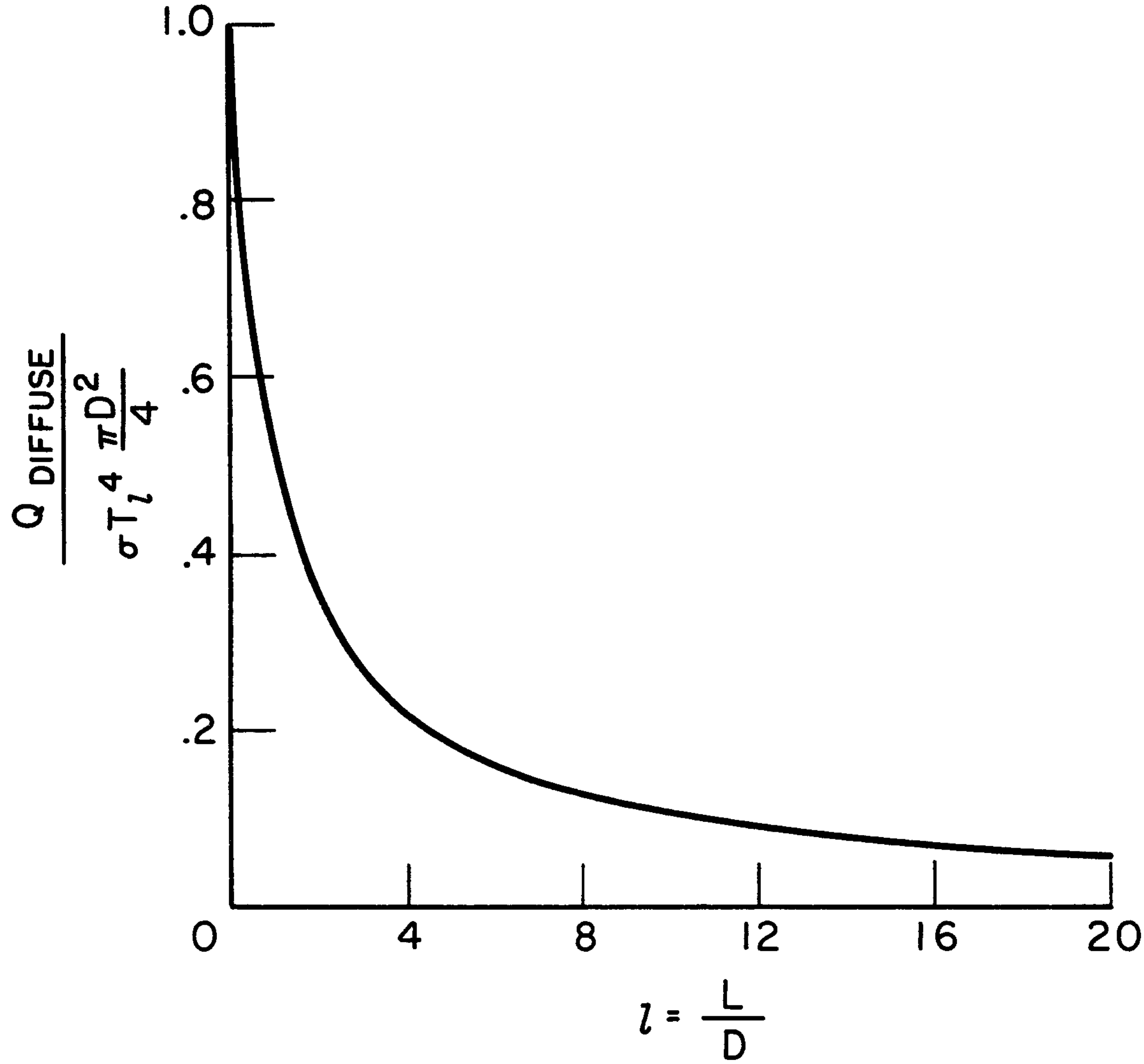

(a) BLACK OR DIFFUSE REFLECTING WALL.

Figure 7. - Net heat transmitted through an unheated tube. $T_l = T_l$; $T_r = 0$.

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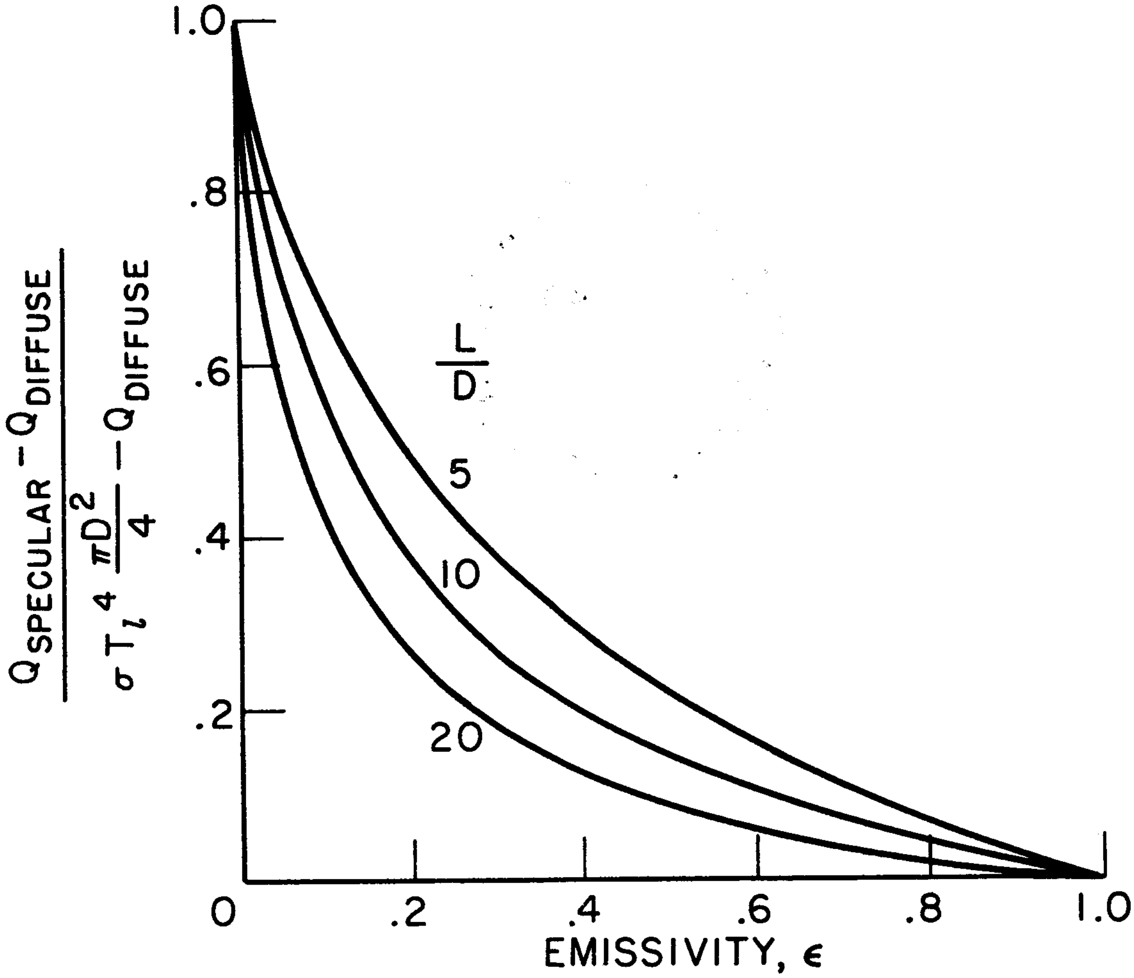

(b) SPECULAR WALL COMPARED WITH DIFFUSE OR BLACK WALL.

Figure 7. - Concluded. Net heat transmitted through an unheated tube. $T_l = T_l$; $T_r = 0$.